

NON-AUTONOMOUS DEGENERATE KdV SYSTEMS

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Abstract

Non-autonomous degenerate KdV systems in $(1+1)$ dimensions are considered for integrability classification. Integrability of the systems is associated with the existence of a recursion operator. Some new non-autonomous degenerate two-component KdV systems are found.

INTRODUCTION

Systems of integrable nonlinear partial differential equations have been constituting one of the main research areas for about two decades. Recently, in [1] we have classified non-autonomous N -component Korteweg-de Vries (N -KdV) systems

$$q_t^i = q_{xxx}^i + s_{jk}^i(t) q^j q_x^k, \quad s_{jk}^i = s_{kj}^i, \quad i, j, k = 1, 2, \dots, N,$$

where dependent variables are $q^i = q^i(x, t)$ and found that the class of systems

$$q_t^i = q_{xxx}^i + \frac{1}{\sqrt{t}} \tilde{s}_{jk}^i q^j q_x^k,$$

where \tilde{s}_{jk}^i are constants, is integrable with the recursion operator

$$\begin{aligned} R_j^i &= t \delta_j^i D^2 + \frac{2}{3} \sqrt{t} \tilde{s}_{jk}^i q^k + \frac{1}{3} \delta_j^i x + \left(\frac{1}{3} \sqrt{t} \tilde{s}_{jk}^i q_x^k + \frac{1}{6} \delta_j^i \right) D^{-1} \\ &\quad + \frac{1}{9} F_{lkj}^i q^l D^{-1} q^k D^{-1}, \end{aligned}$$

where D is the total x -derivative and D^{-1} is the –formal– inverse operator ($DD^{-1} = D^{-1}D = 1$), provided that the identities

$$\tilde{s}_{pr}^k F_{ljk}^i + \tilde{s}_{jr}^k F_{lpk}^i + \tilde{s}_{jp}^k F_{lrk}^i = 0, \quad (1)$$

where

$$F_{plj}^i = \tilde{s}_{jk}^i \tilde{s}_{lp}^k - \tilde{s}_{lk}^i \tilde{s}_{jp}^k,$$

are satisfied. Following Svinolupov's discovery that constants \tilde{s}_{jk}^i satisfying (1) are associated with structure constants of (commutative, non-associative) Jordan algebras [2, 3], we called these systems as Non-autonomous Svinolupov Jordan KdV Systems. Moreover, we have also shown that systems associated with a certain type of Jordan algebras are not transformable to autonomous ones.

In the present work, we consider a system of N equations

$$q_t^i = b_j^i q_{xxx}^j + s_{jk}^i(t) q^j q_x^k + y_j^i(t) q^j, \quad i, j, k = 1, 2, \dots, N, \quad (2)$$

and associate integrability of this system with the existence of a recursion operator whose form we assume to be

$$R_j^i = z_j^i(t)D^2 + a_{jk}^i(t)q^k + h_j^i(x, t) + (c_{jk}^i(t)q_x^k + w_j^i(t))D^{-1}. \quad (3)$$

Here, in (2) and (3) except for b_j^i which are assumed to be constants, all the coefficient terms are introduced with their presumed dependence on the independent variables x and t . Sufficient differentiability of these coefficients with respect to the independent variables is the other assumption made.

We first present the conditions due to the integrability criteria [4]

$$R_{j,t}^i = K_r'^i R_j^r - R_r^i K_j'^r, \quad (4)$$

where $K_j'^i = b_j^i D^3 + s_{kj}^i(t)q^k D + s_{jk}^i(t)q_x^k + y_j^i(t)$ is the Fréchet derivative of system (2). And then, we find solutions of these conditions for degenerate systems, i.e systems having b_j^i such that $|b_j^i| = 0$, in $N = 2$.

INTEGRABILITY CONDITIONS

The integrability criteria (4) leads to some algebraic and differential conditions among the coefficient terms of the system (2) and the recursion operator (3). In the following proposition we present these conditions.

Proposition 1: Let $q^i(x, t)$ be functions of x and t satisfying equations (2) and admitting a recursion operator R_j^i given in (3). Then the coefficient terms $b_j^i, s_{jk}^i(t), y_j^i(t), z_j^i(t), a_{jk}^i(t), h_j^i(x, t), c_{jk}^i(t), w_j^i(t)$ satisfy the following relations:

$$\begin{aligned} 3b_k^i a_{jl}^k + b_k^i c_{jl}^k + s_{kl}^i z_j^k - 2z_k^i s_{lj}^k - z_k^i s_{jl}^k - c_{kl}^i b_j^k &= 0, \quad s_{mk}^i a_{jl}^k - a_{km}^i s_{lj}^k = 0, \\ 3b_k^i a_{jl}^k + 3b_k^i c_{jl}^k - z_k^i s_{lj}^k - 2z_k^i s_{jl}^k &= 0, \quad a_{jk}^i b_l^k - b_k^i a_{jl}^k - 3b_k^i c_{jl}^k + z_k^i s_{jl}^k = 0, \\ b_k^i z_j^k - z_k^i b_j^k &= 0, \quad b_k^i h_j^k - h_k^i b_j^k = 0, \quad b_k^i c_{jl}^k - c_{jk}^i b_l^k = 0, \quad w_k^i (s_{lj}^k - s_{jl}^k) = 0, \\ c_{km}^i (s_{jl}^k - s_{lj}^k) &= 0, \quad c_{jk}^i s_{ml}^k - s_{km}^i c_{jl}^k = 0, \quad c_{jk}^i s_{ml}^k - s_{mk}^i c_{jl}^k = 0, \\ a_{jk}^i s_{ml}^k + a_{km}^i s_{jl}^k + c_{kl}^i s_{jm}^k - s_{mk}^i a_{jl}^k - s_{mk}^i c_{jl}^k - s_{kl}^i a_{jm}^k &= 0, \quad (5) \\ b_k^i a_{jl}^k + s_{lk}^i z_j^k - z_k^i s_{lj}^k - a_{kl}^i b_j^k &= 0, \quad s_{lk}^i h_j^k - h_k^i s_{lj}^k = 0, \quad s_{kl}^i h_j^k - h_k^i s_{jl}^k = 0, \\ z_{j,t}^i - 3b_k^i h_{j,x}^k - b_k^i w_j^k - y_k^i z_j^k + z_k^i y_j^k + w_k^i b_j^k &= 0, \quad w_{j,t}^i - y_k^i w_j^k + w_k^i y_j^k = 0, \end{aligned}$$

$$\begin{aligned}
a_{jl,t}^i + a_{jk}^i y_l^k - s_{lk}^i h_{j,x}^k - s_{lk}^i w_j^k - y_k^i a_{jl}^k + a_{kl}^i y_j^k + w_k^i s_{jl}^k &= 0, \quad b_k^i h_{j,2x}^k = 0, \\
c_{jl,t}^i + c_{jk}^i y_l^k - s_{kl}^i w_j^k - y_k^i c_{jl}^k + c_{kl}^i y_j^k &= 0, \quad h_{j,t}^i - b_k^i h_{j,3x}^k - y_k^i h_j^k + h_k^i y_j^k = 0.
\end{aligned}$$

This proposition is the straight forward result of calculating (4) together with (2) and (3).

Our basic aim is to determine the integrable systems (or classes of systems) with their associated recursion operators by solving the whole system (5). For this purpose we base our classification on b_j^i and solve conditions (5) for each of specially chosen different forms of b_j^i matrix. Having chosen a certain b_j^i , we first solve the algebraic part of the equations in (5). Although the algebraic part is quite large even in $N = 2$, because the constituent equations are polynomials of order at most two in the unknowns, they can be solved by computer algebra systems conveniently. We used REDUCE and MuPAD software for the computations. Each nontrivial solution to the algebraic part is then subjected to the remaining differential conditions.

In general, for a certain b_j^i matrix there are many solutions to the integrability conditions (5). However, some of these solutions give rise to systems which are not valuable: Decoupled systems, trivially coupled systems like $u_t = F[u, t], v_t = G[u, t]$ or completely linear systems. Besides these, some solutions require the recursion operator to be (or proportional to) identity, in case of which (4) is obviously inconclusive for the integrability of the system at hand. These basically constitute our criteria for triviality of a solution to (5). We discard all such trivial cases.

CLASSIFICATION

In this algorithm each different b_j^i matrix cause a different set of solutions to (5). We call a collection of systems obtained under a certain b_j^i as a class and naturally identify each class with its b_j^i matrix. Here, difference of matrices refers to dissimilarity of them.

Since our particular concern is the degenerate 2-KdV systems in this work, we consider the following degenerate and dissimilar forms for b_j^i matrix:

$$b(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b(2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (6)$$

Remark 1: Up to an overall factor, the above two matrix which are in Jordan canonical form constitute all the non-zero degenerate matrices in

$N = 2$ under the similarity transformations. I.e any non-zero, degenerate, 2×2 matrix b_l^k is transformable to one of the above two with an overall factor by an invertible (similarity) transformation M_j^i as $\tilde{b}_j^i = M_k^i b_l^k (M^{-1})_j^l$, where M_j^i are constants. Existence of such invertible M_j^i 's is assured by Jordan canonical form theorem of matrices [5]. The null matrix and the overall factors for $b(1)$ and $b(2)$ are irrelevant for our classification even if these factors are taken as functions of t . They can always be absorbed into the left hand side of evolution system (2) by redefining the evolution parameter t .

Remark 2: The above mentioned similarity transformations correspond to change of dependent variables $\tilde{q}^i = M_k^i q^k$ and the integrability criteria (4) is invariant under such transformations. We do not distinguish a system from its M_j^i transformed form

$$\begin{aligned} M_k^i q_t^k &= M_k^i b_l^k (M^{-1})_j^l M_r^j q_{xxx}^r + M_k^i s_{jl}^k (M^{-1})_r^j (M^{-1})_p^l M_s^r q^s M_\nu^p q_x^\nu \\ &\quad + M_k^i y_j^k (M^{-1})_p^j M_s^p q^s. \end{aligned}$$

Moreover, we suggest to use the form of a KdV system where its b_j^i is put in Jordan canonical form, as the standard form for it.

a) Systems associated with $b_j^i = b(1)$

Although our main concern is the non-autonomous systems, solutions of integrability conditions (5) with $b_j^i = b(1)$ in (6) give rise to autonomous systems as well. In the following list we first give these autonomous systems with their respective recursion operators.

i)

$$\begin{aligned} u_t &= u_{xxx} + 3c_1 u u_x \\ v_t &= c_1 u v_x + c_2 v u_x \end{aligned}, \quad R = \begin{pmatrix} D^2 + 2c_1 u + c_1 u_x D^{-1} & 0 \\ c_2 v + c_1 v_x D^{-1} & 0 \end{pmatrix}, \quad (7)$$

where c_1 and c_2 are arbitrary constants [6],[7].

ii)

$$\begin{aligned} u_t &= u_{xxx} + 3c_1 u u_x + c_2 v v_x \\ v_t &= c_1 (u v)_x \end{aligned}, \quad R = \begin{pmatrix} D^2 + 2c_1 u + c_1 u_x D^{-1} & c_2 v \\ c_1 (v + v_x D^{-1}) & 0 \end{pmatrix}, \quad (8)$$

where c_1 and c_2 are arbitrary constants. This system is equivalent to the Ito system via $u \rightarrow \frac{2}{c_1} u$, $v \rightarrow \frac{2}{\sqrt{c_1 c_2}} v$ transformation [8].

iii)

$$\begin{aligned} u_t &= u_{xxx} + 3c_1uu_x, \\ v_t &= (c_1u + c_2v)v_x \end{aligned}, \quad R = \begin{pmatrix} D^2 + 2c_1u + c_1u_xD^{-1} & 0 \\ c_1v_xD^{-1} & c_2v \end{pmatrix}, \quad (9)$$

where c_1 and c_2 are arbitrary constants.

At this point, for the above list of systems we have the following statement.

Proposition 2: Systems (7), (8) and (9) constitute all the nontrivial autonomous systems in $b(1)$ class.

The followings are the non-autonomous systems with their respective recursion operators in $b(1)$ class.

i)

$$\begin{aligned} u_t &= u_{xxx} + c_1t^{-(\alpha+2/3)}vv_x + c_2t^{-(\alpha+1)}v, \\ v_t &= t^{-2/3}vv_x, \end{aligned} \quad (10)$$

where c_1, c_2 and α are arbitrary constants. This system admits the recursion operator

$$R = \begin{pmatrix} tD^2 + \frac{x}{3} + \alpha D^{-1} & t^{-\alpha}(c_1t^{1/3}v + c_2D^{-1}) \\ 0 & t^{1/3}v + \frac{x}{3} \end{pmatrix}. \quad (11)$$

ii)

$$\begin{aligned} u_t &= u_{xxx} + e^{-\alpha t}(c_1vv_x + c_2v), \\ v_t &= vv_x, \end{aligned} \quad (12)$$

where c_1, c_2 and α are arbitrary constants. The recursion operator is

$$R = \begin{pmatrix} D^2 + \alpha D^{-1} & e^{-\alpha t}(c_1v + c_2D^{-1}) \\ 0 & v \end{pmatrix}. \quad (13)$$

iii)

$$\begin{aligned} u_t &= u_{xxx} + t^{-1/2}uu_x, \\ v_t &= \frac{1}{3}(t^{-1/2}uv_x + t^{-2/3}vv_x + t^{-5/6}u). \end{aligned} \quad (14)$$

This system admits the recursion operator $R = \begin{pmatrix} R_0^0 & R_1^0 \\ R_0^1 & R_1^1 \end{pmatrix}$ with

$$\begin{aligned} R_0^0 &= tD^2 + \frac{x}{3} + \frac{2}{3}t^{1/2}u + \frac{1}{3}(t^{1/2}u_x + \frac{1}{2})D^{-1}, \\ R_1^0 &= 0, \\ R_0^1 &= \frac{1}{3}(t^{1/2}v_x + t^{1/6})D^{-1}, \\ R_1^1 &= \frac{1}{3}(t^{1/3}v + x). \end{aligned} \quad (15)$$

On the completeness of the above given non-autonomous systems we have the following proposition.

Proposition 3: Systems (10), (12) and (14) constitute all the nontrivial, integrable non-autonomous systems in $b(1)$ class.

All other solutions of (5) are trivial. Explicit solutions of (5) with $b_j^i = b(1)$ prove propositions 2 and 3.

Systems related by invertible transformations can be considered equivalent [1, 9, 10, 11]. For this reason, we analyzed the above mentioned non-autonomous systems in regard to their transformability to autonomous systems via a certain generic class of transformations. In the next proposition we present the result.

Proposition 4: None of the non-autonomous systems (10), (12) or (14) is transformable to any autonomous system up to the invertible change of variables

$$\begin{aligned} x &= \alpha(\tilde{t})\tilde{x} + \beta(\tilde{t}), \quad t = \gamma(\tilde{t}), \\ u(x, t) &= \delta(\tilde{t})\tilde{u}(\tilde{x}, \tilde{t}) + \phi(\tilde{x}, \tilde{t}), \\ v(x, t) &= \rho(\tilde{t})\tilde{v}(\tilde{x}, \tilde{t}) + \psi(\tilde{x}, \tilde{t}). \end{aligned} \quad (16)$$

The $\alpha = 0$ special case of system (12) is the only exception. In this case (12) is apparently autonomous. This proposition can be verified by direct calculation for each of the mentioned systems.

b) Systems associated with $b_j^i = b(2)$

Solutions of integrability conditions (5) under $b_j^i = b(2)$ in (6) give rise to the following nontrivial, integrable subclasses of systems.

i)

$$\begin{aligned} u_t &= v_{xxx} + vu_x + c_1uv_x + svv_x + yv, \\ v_t &= vv_x, \end{aligned} \quad (17)$$

where c_1 is an arbitrary constant, is integrable if the undetermined functions $s = s(t)$ and $y = y(t)$ satisfy the differential constraint

$$\frac{ds}{dt} + c_1y = 0. \quad (18)$$

These systems admit the recursion operator

$$R = \begin{pmatrix} (1 - \frac{c_1}{2})v & D^2 + c_1u + sv + \frac{c_1}{2}u_xD^{-1} \\ 0 & (1 - \frac{c_1}{2})v + \frac{c_1}{2}v_xD^{-1} \end{pmatrix}. \quad (19)$$

ii)

$$\begin{aligned} u_t &= v_{xxx} + c_1(vu_x + 2uv_x) + svv_x + yv, \\ v_t &= vv_x, \end{aligned} \quad (20)$$

where c_1 is an arbitrary constant, is integrable if the undetermined functions $s = s(t)$ and $y = y(t)$ satisfy the differential constraint

$$\frac{ds}{dt} + (3c_1 - 1)y = 0. \quad (21)$$

The recursion operator is

$$R = \begin{pmatrix} 0 & D^2 + 2c_1u + sv + c_1u_xD^{-1} \\ 0 & (1 - c_1)v + c_1v_xD^{-1} \end{pmatrix}. \quad (22)$$

In this subclass we observed that the $c_1 = 1$ particular case of (20) with (21) admits two recursion operators $R(1)$ and $R(2)$. The first one being (22) with $c_1 = 1$ and the second one is

$$R(2) = \begin{pmatrix} 0 & tD^2 + 2tu + tsv + (tu_x - \frac{1}{2}s)D^{-1} \\ 0 & (tv_x + 1)D^{-1} \end{pmatrix}. \quad (23)$$

iii)

$$\begin{aligned} u_t &= v_{xxx} + t^{-2/3}(vu_x + c_1vv_x) + yv, \\ v_t &= t^{-2/3}vv_x, \end{aligned} \quad (24)$$

where c_1 is an arbitrary constant, is integrable for any arbitrary function $y = y(t)$. The recursion operators of this system are

$$R(1) = \begin{pmatrix} \frac{x}{3} + t^{1/3}v & tD^2 + c_1t^{1/3}v \\ 0 & \frac{x}{3} + t^{1/3}v \end{pmatrix}, \quad R(2) = \begin{pmatrix} 0 & D^{-1} \\ 0 & 0 \end{pmatrix}. \quad (25)$$

The second operator $R(2)$ is not a proper recursion operator because of its nilpotent character. Nevertheless, $R(2)$ itself or any linear combination of it with $R(1)$ satisfies the conditions to be a recursion operator for (24).

iv)

$$\begin{aligned} u_t &= v_{xxx} + t^\alpha \left(uv_x + \frac{\alpha+1}{3\alpha+2}vu_x + svv_x \right) + yv, \\ v_t &= \frac{\alpha+1}{3\alpha+2}t^\alpha vv_x, \end{aligned} \quad (26)$$

where $\alpha \neq -2/3$ is an arbitrary constant, is integrable if the undetermined functions $s = s(t)$ and $y = y(t)$ satisfy the differential constraint

$$\frac{ds}{dt} + y = 0. \quad (27)$$

These systems admit the recursion operator

$$\begin{aligned} R_0^0 &= -\frac{\alpha}{2} \left(x + \frac{t^{\alpha+1}}{3\alpha+2} v \right), \\ R_1^0 &= tD^2 + t^{\alpha+1}u + t^{\alpha+1}sv + \frac{1}{2} (t^{\alpha+1}u_x - (3\alpha+2)s) D^{-1}, \\ R_0^1 &= 0, \\ R_1^1 &= -\frac{\alpha}{2} \left(x + \frac{t^{\alpha+1}}{3\alpha+2} v \right) + \frac{1}{2} (t^{\alpha+1}v_x + 3\alpha+2) D^{-1}. \end{aligned} \quad (28)$$

v)

$$\begin{aligned} u_t &= v_{xxx} + e^{t/\alpha} (vu_x + 3uv_x + svv_x) + yv, \\ v_t &= e^{t/\alpha} vv_x, \end{aligned} \quad (29)$$

where $\alpha \neq 0$ is an arbitrary constant, is integrable if the undetermined functions $s = s(t)$ and $y = y(t)$ satisfy the differential constraint

$$\frac{ds}{dt} + 3y = 0. \quad (30)$$

The recursion operator is

$$\begin{aligned} R_0^0 &= -\frac{1}{2} (x + \alpha e^{t/\alpha} v), \\ R_1^0 &= \alpha \left[D^2 + 3e^{t/\alpha} u + e^{t/\alpha} sv + \frac{1}{2} (3e^{t/\alpha} u_x - \frac{1}{\alpha} s) D^{-1} \right], \\ R_0^1 &= 0, \\ R_1^1 &= -\frac{1}{2} (x + \alpha e^{t/\alpha} v) + \frac{3}{2} (\alpha e^{t/\alpha} v_x + 1) D^{-1}. \end{aligned} \quad (31)$$

vi)

$$\begin{aligned} u_t &= v_{xxx} + (uv)_x + svv_x + yv, \\ v_t &= vv_x, \end{aligned} \quad (32)$$

is integrable for any arbitrary functions $s = s(t)$ and $y = y(t)$. These systems admit the recursion operators $R(1)$, $R(2)$ and $R(3)$ where

$$\begin{aligned} R(1)_0^0 &= -v_x D^{-1}, \\ R(1)_1^0 &= D^2 + 2u - 2(\int y dt)v + [u_x + (\int y dt)v_x] D^{-1}, \\ R(1)_0^1 &= 0, \\ R(1)_1^1 &= v_x D^{-1}, \end{aligned} \quad (33)$$

$$\begin{aligned}
R(2)_0^0 &= -x - tv - 2(tv_x + 1)D^{-1}, \\
R(2)_1^0 &= 2tu + 2\left(\int (s - ty)dt\right)v \\
&\quad + \left[tu_x + \left(\int (s + 2ty + 3\int ydt)dt\right)v_x + 3\int ydt\right]D^{-1}, \\
R(2)_0^1 &= 0, \\
R(2)_1^1 &= -x - tv + (tv_x + 1)D^{-1},
\end{aligned} \tag{34}$$

$$R(3) = \begin{pmatrix} 0 & x + tv \\ 0 & 0 \end{pmatrix}. \tag{35}$$

Obviously, $R(3)$ is nilpotent and thus it is not a proper recursion operator.
vii)

$$\begin{aligned}
u_t &= v_{xxx} + \frac{1}{\sqrt{t}}(uv)_x + svv_x + yv, \\
v_t &= \frac{1}{\sqrt{t}}vv_x,
\end{aligned} \tag{36}$$

is integrable for any arbitrary functions $s = s(t)$ and $y = y(t)$. These systems admit the recursion operators $R(1)$, $R(2)$ and $R(3)$ where

$$\begin{aligned}
R(1)_0^0 &= x + 2\sqrt{t}v + (2\sqrt{t}v_x + 1)D^{-1}, \\
R(1)_1^0 &= 2tD^2 - \left[\left(\int (2\sqrt{t}y + \frac{1}{\sqrt{t}}\int ydt)dt\right)v_x + \int ydt\right]D^{-1}, \\
R(1)_0^1 &= 0, \\
R(1)_1^1 &= x + 2\sqrt{t}v,
\end{aligned} \tag{37}$$

$$\begin{aligned}
R(2)_0^0 &= x + 2\sqrt{t}v, \\
R(2)_1^0 &= 4tD^2 + 4\sqrt{t}u + 2\left(\int (s - 2\sqrt{t}y)dt\right)v \\
&\quad + \left[2\sqrt{t}u_x + \left(\int (s + \frac{1}{\sqrt{t}}\int ydt)dt\right)v_x + \int ydt\right]D^{-1}, \\
R(2)_0^1 &= 0, \\
R(2)_1^1 &= x + 2\sqrt{t}v + (2\sqrt{t}v_x + 1)D^{-1},
\end{aligned} \tag{38}$$

$$R(3) = \begin{pmatrix} 0 & x + 2\sqrt{t}v \\ 0 & 0 \end{pmatrix}. \tag{39}$$

Again we obtain an improper recursion operator $R(3)$.

There are other solutions to integrability conditions (5) in $b(2)$ class as well. However, those solutions correspond to trivial systems or recursion operators. For this reason we have the following proposition.

Proposition 4: The subclasses (20)-(36) constitute all the nontrivial subclasses in $b(2)$ class.

About the transformability of non-autonomous systems in the above subclasses, we have the following proposition.

Proposition 5: Non-autonomous systems in each of the subclasses (20)-(36) of $b(2)$ class are not transformable to any autonomous system through the transformations (16).

It has been recently observed that some recursion operators which are called weak, do not always generate symmetry hierarchies correctly [12]. Some of the operators we found in this work are of this type. The source of this weakness and possible solutions which relies on the found weak recursion operators are extensively investigated in [13].

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